LESSON

10 Rational Exponents

UNDERSTAND An exponential expression includes a **base** raised to an **exponent**, or **power.** The properties of exponents can help you simplify many exponential expressions and solve equations involving exponents. Some of those properties are listed below.

Product of powers:	$a^n \cdot a^m = a^{n+m}$
Power of a product:	$(ab)^m = a^m b^m$
Power of a power:	$(a^n)^m = a^{n \cdot m}$
Power of zero:	$a^0 = 1$ for all $a \neq 0$
Quotient of powers:	$\frac{a^m}{a^n} = a^{m-n}$ for all $a \neq 0$
Power of a quotient:	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ for all $b \neq 0$
Negative powers:	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$ for all $a \neq 0$

An exponential expression can be evaluated for any **rational exponent**. Until now, you have worked primarily with integer powers, but sometimes you may need to simplify or evaluate an exponential expression for other powers. The properties of exponents can help you rewrite expressions with fractional exponents in a more familiar form.

You know that $2 \cdot \frac{1}{2} = 1$, $3^1 = 3$, and $\sqrt{9} = 3$. By applying the substitution property of equality and the power of a power property, you can find an equivalent form of a fractional exponent.

$3^1 = 3$	Substitute $2 \cdot \frac{1}{2}$ for 1.
$3^{2 \cdot \frac{1}{2}} = 3$	Apply the power of a power property.
$(3^2)^{\frac{1}{2}} = 3$	Evaluate inside the parentheses. Substitute $\sqrt{9}$ for 3.
$9^{\frac{1}{2}} = \sqrt{9}$	

Raising a number to the power $\frac{1}{2}$ is equivalent to taking its square root.

In general, an exponential expression with a fractional exponent involves a **root.** In converting between the exponential and **radical** forms, the base becomes the **radicand**, the denominator of the fraction becomes the **index** of the root, and the numerator of the fraction becomes an integer exponent for the expression.

A base a with exponent $\frac{1}{n}$ is the same as the nth root of the number a.

$a^{\frac{1}{n}} = \sqrt[n]{a}$ $\qquad 11^{\frac{1}{3}} = \sqrt[3]{11}$

A base a with exponent $\frac{m}{n}$ is the same as the nth root of the number a raised to the mth power.

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ $\qquad 11^{\frac{5}{2}} = (\sqrt{11})^5 = \sqrt{11^5}$

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Connect

Simplify the expression $\frac{\sqrt{x} \cdot \sqrt[3]{x}}{\sqrt[6]{x^5}} + x^{\frac{3}{2}}$.

1 Rewrite each radical expression using exponents.

In the expression $\sqrt{x}$, the unwritten index is 2, so $\sqrt{x} = x^{\frac{1}{2}}$.

In the expression $\sqrt[3]{x}$, the index is 3, so $\sqrt[3]{x} = x^{\frac{1}{3}}$.

In the expression $\sqrt[6]{x^5}$, the index is 6 and the exponent is 5, so $\sqrt[6]{x^5} = x^{\frac{5}{6}}$.

Now, rewrite the expression with no radicals.

$$\frac{\sqrt{x} \cdot \sqrt[3]{x}}{\sqrt[6]{x^5}} + x^{\frac{3}{2}} = \frac{x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}}{x^{\frac{5}{6}}} + x^{\frac{3}{2}}$$

2 Simplify the numerator of the fraction.

Use the product of powers property to multiply the terms.

$x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}$

$x^{\frac{1}{2} + \frac{1}{3}}$

$x^{\frac{3}{6} + \frac{2}{6}}$

$x^{\frac{5}{6}}$

Now, rewrite the expression.

$$\frac{x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}}{x^{\frac{5}{6}}} + x^{\frac{3}{2}} = \frac{x^{\frac{5}{6}}}{x^{\frac{5}{6}}} + x^{\frac{3}{2}}$$

3 Simplify the fraction.

Notice that the numerator and denominator of the fraction are identical. Any fraction with the same numerator and denominator is equal to 1.

$$\frac{x^{\frac{5}{6}}}{x^{\frac{5}{6}}} + x^{\frac{3}{2}} = 1 + x^{\frac{3}{2}}$$

4 Rewrite the exponential expression as a radical.

To rewrite the expression $x^{\frac{3}{2}}$ as a radical, the base, x, will be the radicand, the numerator of the fraction, 3, will be an exponent, and the denominator, 2, will be the index of the radical.

▶ $1 + x^{\frac{3}{2}} = 1 + (\sqrt{x})^3$

Since the expression has a rational exponent that is an improper fraction, it can be written in another way.

$$x^{\frac{3}{2}} = x^{1 + \frac{1}{2}} = (x^1)(x^{\frac{1}{2}}) = x\sqrt{x}$$

▶ $1 + x^{\frac{3}{2}} = 1 + x\sqrt{x}$

TRY

Simplify.

$$\left(5^{\frac{1}{c}}\right)^d \cdot \frac{1}{5^{-\frac{d}{c}}}$$

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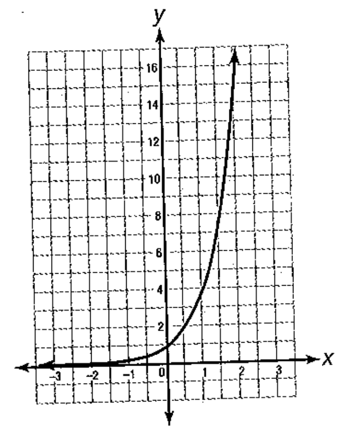
EXAMPLE A Use a table to graph the equation $y = 4^x$. Then, use the graph to confirm values of y for fractional values of x.

1 Make a table of values.

x	$y = 4^x$	y
-1	$y = 4^{-1} = \frac{1}{4}$	$\frac{1}{4}$
0	$y = 4^0 = 1$	1
1	$y = 4^1 = 4$	4
2	$y = 4^2 = 16$	16

2 Graph the equation using the values from the table.



3 Make a table of values with fractional values of x.

x	$y = 4^x$	y
$\frac{1}{4}$	$y = 4^{\frac{1}{4}} = (2^2)^{\frac{1}{4}} = 2^{\frac{2}{4}} = \sqrt{2} \approx 1.414$	1.414
$\frac{1}{3}$	$y = 4^{\frac{1}{3}} = \sqrt[3]{4} \approx 1.587$	1.587
$\frac{1}{2}$	$y = 4^{\frac{1}{2}} = \sqrt{4} = 2$	2
$\frac{3}{2}$	$y = 4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8$	8

4 Compare the values in the table to the graph.

According to the equation, when $x = \frac{1}{4}$ and $x = \frac{1}{3}$, y is close to $1\frac{1}{2}$. The graph passes near $1\frac{1}{2}$ for these values of x.

When $x = \frac{1}{2}$, $y = 2$. The graph has a y-value of 2 halfway between the x-values of 0 and 1.

When $x = \frac{3}{2}$, $y = 8$. The graph has a y-value of 8 halfway between the x-values of 1 and 2.

▶ The values of the equation for fractional values of x match the graph above.

DISCUSS

Graph the equation on your graphing calculator and use the TRACE or TABLE function to confirm your answers. Are your calculations for $x = \frac{1}{4}$ and $x = \frac{1}{3}$ more or less accurate than those on the calculator?

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EXAMPLE B Solve the equation $3^{\frac{x}{2}} = 9\sqrt{3}$ for x.

1 Write all terms as exponential expressions with the same base.

The left side of the equation has a base of 3, so rewrite the expression $9\sqrt{3}$ as an exponential expression with a base of 3.

$9\sqrt{3}$	Substitute 3^2 for 9.
$3^2 \cdot \sqrt{3}$	Rewrite the radical as an exponential expression.
$3^2 \cdot 3^{\frac{1}{2}}$	Use the product of powers property.
$3^{2+\frac{1}{2}}$	Find a common denominator and add.
$3^{\frac{5}{2}}$	

2 Solve the equation.

Substitute the expression you found above for the right side of the equation.

$3^{\frac{x}{2}} = 9\sqrt{3}$	Substitute $3^{\frac{5}{2}}$ for $9\sqrt{3}$.
$3^{\frac{x}{2}} = 3^{\frac{5}{2}}$	Since the bases are equal, set the exponents equal to each other.
$\frac{x}{2} = \frac{5}{2}$	Multiply both sides of the equation by 2.
$x = 5$	

▶ The solution to the equation is $x = 5$.

TRY

Solve the equation $64^y = 16$ for y.

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LESSON

31 Polynomials

UNDERSTAND A **polynomial** consists of constants and variables joined together by addition, subtraction, and/or multiplication. The constants and variables are grouped into one or more terms, each of which can be an individual number, a single variable, or a product of numbers and/or variables with exponents that are non-negative integers.

The polynomial $6ab + 9 - 14c^{10}$ has three terms. The first term, $6ab$, is joined to the second term, 9, through addition. The third term, $14c^{10}$, is joined to the other terms through subtraction. Or you could say that the term $-14c^{10}$ is added to the other terms.

The expression $200 \cdot 7^t - 1$ is not a polynomial because the first term contains a variable, t, in the exponent. This is an exponential expression. The expression $7fg^{-1}$ is not a polynomial because it has a negative exponent. This is equal to $\frac{7f}{g}$, which is a rational expression.

UNDERSTAND Polynomials with 1, 2, or 3 terms can be grouped into categories.

- A **monomial** is a polynomial having only 1 term, such as $12xy^3$.
- A **binomial** is a polynomial having exactly 2 terms, such as $y^2 + 4$.
- A **trinomial** is a polynomial having exactly 3 terms, such as $4x^4 - \frac{1}{6}x + 11$.

Polynomials can also be categorized according to their **degree**. The degree of a monomial with one variable is equal to the value of the variable's exponent. The monomial $5t^3$ has degree 3. The degree of a polynomial is equal to the highest degree of its terms. The three terms of the polynomial $x^5 - 3x^2 + 4$ have degrees 5, 2, and 0, respectively, so this polynomial is of degree 5.

UNDERSTAND The standard form of a polynomial contains no like terms. For example, the expression $2x^2 + 3x - x - 7$ simplifies to $2x^2 + 2x - 7$. The like terms, $3x$ and $-x$, can be combined by using the distributive property.

If a polynomial has more than one degree, express it in standard form by writing its terms in descending order of degree. In other words, the exponents should go from greatest to least. For example, the polynomial $4 + 5a^3 - 2a^6 - 3a$ written in standard form is $-2a^6 + 5a^3 - 3a + 4$.

UNDERSTAND The set of integers is a collection of every integer number. You know that whenever you add or subtract two integers, the result is also an integer. The product of any two integers is also an integer. The set of integers is said to be closed under addition, subtraction, and multiplication.

The set of polynomials includes every possible polynomial. When any two polynomials are added or subtracted, the result is always a polynomial. Multiplying two polynomials together also always produces a polynomial. Like the set of integers, the set of polynomials is closed under addition, subtraction, and multiplication.

Connect

Add the polynomials and write the sum in standard form: $(9x - 2x^2 + 13) + (7x^2 + 1 - 3x)$.

1

Use properties of addition to group like terms together.

$(9x - 2x^2 + 13) + (7x^2 + 1 - 3x)$	Use the associative property to remove parentheses.
$9x - 2x^2 + 13 + 7x^2 + 1 - 3x$	Use the commutative property to reorder the terms.
$9x - 3x - 2x^2 + 7x^2 + 13 + 1$	Use the associative property to insert parentheses.
$(9x - 3x) + (-2x^2 + 7x^2) + (13 + 1)$	

2

Combine like terms.

Use the distributive property to combine like terms.

$(9x - 3x) + (-2x^2 + 7x^2) + (13 + 1)$

$(9 - 3)x + (-2 + 7)x^2 + (13 + 1)$

$6x + 5x^2 + 14$

3

Rewrite the result in standard form.

The term with the highest degree is $5x^2$. The term with the next-highest degree is $6x$. The constant term will be last.

▶ The sum is the polynomial $5x^2 + 6x + 14$.

Find the difference of the polynomials and write your answer in standard form: $(4a^2 - 6a + 2) - (a^2 - 5)$.

1

Group like terms together.

Since the second polynomial is being subtracted, be sure to distribute the negative sign to both terms in the binomial.

$(4a^2 - 6a + 2) - (a^2 - 5)$

$4a^2 - 6a + 2 - a^2 + 5$

$4a^2 - a^2 - 6a + 2 + 5$

$(4a^2 - a^2) - 6a + (2 + 5)$

2

Combine like terms.

$(4a^2 - a^2) - 6a + (2 + 5)$
$3a^2 - 6a + 7$

Note that this is already in standard form.

▶ The difference is the polynomial $3a^2 - 6a + 7$.

TRY

Find the difference.
$(6b^3 + b^2 + 14) - (-3b^3 - 9b)$

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EXAMPLE A Find the product: $(2z + 3)(8z^4 + 3z + 7)$.

1 Apply the distributive property.

You can distribute the trinomial to each term in the binomial.

$(2z + 3)(8z^4 + 3z + 7)$

$2z(8z^4 + 3z + 7) + 3(8z^4 + 3z + 7)$

2 Multiply the first term in the binomial by each term in the trinomial.

Distribute $2z$ to each term in the trinomial. Remember: When multiplying exponential terms with the same base, add the exponents.

$2z(8z^4 + 3z + 7) + 3(8z^4 + 3z + 7)$

$2z(8z^4) + 2z(3z) + 2z(7) + 3(8z^4 + 3z + 7)$

$16z^5 + 6z^2 + 14z + 3(8z^4 + 3z + 7)$

3 Multiply the second term in the binomial by each term in the trinomial.

Distribute 3 to each term in the trinomial.

$16z^5 + 6z^2 + 14z + 3(8z^4 + 3z + 7)$

$16z^5 + 6z^2 + 14z + 3(8z^4) + 3(3z) + 3(7)$

$16z^5 + 6z^2 + 14z + 24z^4 + 9z + 21$

4 Combine like terms.

Reorder the terms so that they are in descending order of degree. Place like terms together.

$16z^5 + 6z^2 + 14z + 24z^4 + 9z + 21$

$16z^5 + 24z^4 + 6z^2 + (14z + 9z) + 21$

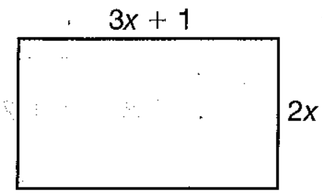
$16z^5 + 24z^4 + 6z^2 + 23z - 21$

▶ The product is the polynomial $16z^5 + 24z^4 + 6z^2 + 23z - 21$.

DISCUSS

The set of polynomials is closed under addition, subtraction, and multiplication. This means that the sum, difference, and product of polynomials is always a polynomial. Is the set closed under division? Consider the example $(3x^2 + 1) \div x$.

EXAMPLE B The rectangle shown below has length $3x + 1$ and width $2x$. Find expressions for its perimeter and area, and confirm that each is a polynomial.



1 Find the perimeter of the rectangle.

The formula for the perimeter of a rectangle is $P = 2l + 2w$. Substitute the expressions for length and width into the formula and simplify.

$P = 2(3x + 1) + 2(2x)$

$P = 6x + 2 + 4x$

$P = 10x + 2$

2 Determine whether the expression for the perimeter is a polynomial.

The expression $10x + 2$ consists of constants and variables joined only through multiplication and addition. The exponent of the variable, x, is a nonnegative integer, 1.

So, $10x + 2$ is a polynomial.

3 Find the area of the rectangle.

The formula for the area of a rectangle is $A = lw$. Substitute the expressions for length and width into the formula and simplify.

$A = (3x + 1)(2x)$

$A = (3x \cdot 2x) + (1 \cdot 2x)$

$A = 6x^2 + 2x$

4 Determine whether the expression for the area is a polynomial.

The expression $6x^2 + 2x$ consists of constants and variables joined only through multiplication and addition. The variable, x, has two nonnegative integer exponents, 2 and 1.

So, $6x^2 + 2x$ is a polynomial.

TRY

A rectangle has area $3x$ and width x^2. Will the expression for the length of the rectangle be a polynomial?

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LESSON 33

Writing Equivalent Polynomial and Exponential Equations

Factoring Quadratic Expressions

UNDERSTAND Multiplying two linear binomial expressions results in a quadratic expression.

$$(2x + 1)(3x - 5) = 6x^2 - 7x - 5$$

Notice that the product of these two binomials simplified to a trinomial.

UNDERSTAND Multiplying two numbers yields a product. The reverse of this process is separating a number into factors. For example, the number 20 can be separated into $4 \cdot 5$. The **prime factorization** of a number is the string of **prime numbers** which, when multiplied, yield that number. The prime factorization of 20 is $2^2 \cdot 5$.

Polynomials also have prime factorizations. A polynomial is prime if it cannot be factored into two other polynomials. The polynomial $6x^2 - 7x - 5$ has the prime factorization $(2x + 1)(3x - 5)$ because both of these binomial factors cannot be factored further. The factors of a polynomial can be numbers, variables, or expressions. There are several methods of factoring polynomials. Examples in this lesson will demonstrate some of these methods. When factoring on your own, choose the method with which you are most comfortable.

The first step in factoring is always to factor out any numbers or variables common to all terms. To do so, find the greatest common factor (GCF) of all of the terms. The terms in the expression below have a GCF of $2x$.

$$12x^3 - 14x^2 - 10x = 2x(6x^2 - 7x - 5)$$

From the multiplication performed at the top of this page, you know that $(2x + 1)$ and $(3x - 5)$ are the factors of $6x^2 - 7x - 5$.

$$2(6x^2 - 7x - 5) = 2(2x + 1)(3x - 5)$$

Neither of these binomials can be factored further, so this is the prime factorization of the polynomial. Not all trinomials can be factored. For example, $x^2 + x + 1$ is a prime polynomial.

UNDERSTAND If you memorize the forms of some special polynomials, you can write their factorizations without going through the steps above.

	Polynomial	Factorization
Square of a sum	$a^2 + 2ab + b^2$	$(a + b)^2$
Square of a difference	$a^2 - 2ab + b^2$	$(a - b)^2$
Difference of squares	$a^2 - b^2$	$(a + b)(a - b)$
Sum of cubes	$a^3 + b^3$	$(a + b)(a^2 - ab + b^2)$
Difference of cubes	$a^3 - b^3$	$(a - b)(a^2 + ab + b^2)$

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Connect

Factor the quadratic expression $8x^2 - 20x - 12$ into its prime factors.

1 Factor out any constant and/or variable factors common to all of the terms.

Each term is evenly divisible by 4.

$$8x^2 - 20x - 12 = 4(2x^2 - 5x - 3)$$

The trinomial $2x^2 - 5x - 3$ has the form $ax^2 + bx + c$ with $a = 2$, $b = -5$, and $c = -3$. Factor this trinomial, if possible.

2 Find the factors of a and of c.

The prime factorization of $2x^2 - 5x - 3$ will have the form $(mx + p)(nx + q)$.

$(mx + p)(nx + q)$

$mnx^2 + mqx + npx + pq$

$mnx^2 + (mq + np)x + pq$

The constants m and n are factors of a. The constants p and q are factors of c.

The quadratic term has the coefficient 2. Find the factor pairs of 2. The possible values of m, n are 1, 2 and 2, 1.

The constant term is -3. The possible values of p, q are $-1, 3$; $1, -3$; $3, -1$ and $-3, 1$.

3 Determine which factor pairs combine to yield b.

The coefficient of the linear term, b, is equivalent to $mq + np$. For this polynomial, $b = -5$. So, find the factor pairs for which $mq + np = -5$. Be careful that you substitute the correct values for p and q.

m, n	***p, q***	***mq + np***
1, 2	−1, 3	$(1)(3) + (2)(-1) = 1$
	1, −3	$(1)(-3) + (2)(1) = -1$
	3, −1	$(1)(-1) + (2)(3) = 5$
	−3, 1	$(1)(1) + (2)(-3) = -5$ ✓

4 Substitute the values for m, n, p, and q.

Based on the table, $m = 1$, $n = 2$, $p = -3$, and $q = 1$. Substitute these into $4(mx + p)(nx + q)$.

$$4(1x - 3)(2x + 1)$$

These binomials cannot be factored any further.

▶ The factored form of $8x^2 - 20x - 12$ is $4(x - 3)(2x + 1)$.

TRY

Fully factor the polynomial $3x^2 - 27x + 60$.

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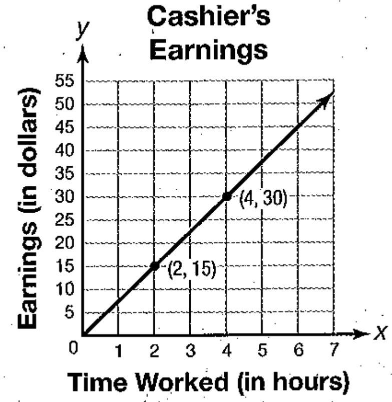
LESSON 13

Average Rate of Change

Finding Average Rate of Change

UNDERSTAND Rates allow us to relate quantities measured in different units. For example, the table and graph below show a linear function that compares the number of hours a cashier works to his total earnings, in dollars.

Cashier's Earnings	
Time in hours, x	**Earnings in \$, y**
0	0
2	15
4	30



The cashier's earnings change, depending on the number of hours he works. His pay rate is an example of a **rate of change**. A rate of change shows how one quantity changes relative to another quantity. To calculate the average rate of change between two ordered pairs (x_1, y_1) and (x_2, y_2), use this formula:

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

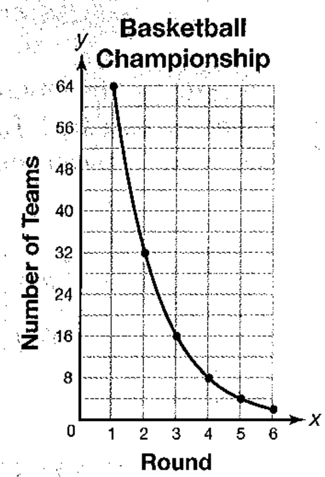
For the function describing the cashier's earnings, choose two ordered pairs, such as (2, 15) and (4, 30).

$$\text{average rate of change} = \frac{30 - 15}{4 - 2} = \frac{15}{2} = 7.50$$

In this case, the rate of change compares dollars earned to hours worked. So, the cashier's rate of pay is \$7.50 per hour.

Connect

A basketball championship begins with 64 teams. Every time a team wins a game, it goes on to the next round. Once a team loses a game, it is eliminated from competition and does not play any more games. The number of teams in each round of the championship is a function of the round. That function is represented on the graph to the right. Compare the rate of change between rounds 1 and 2 to the rate of change between rounds 2 and 3.



1 Calculate the average rate of change between rounds 1 and 2.

Find the rate of change from (1, 64) to (2, 32).

$\frac{32 - 64}{2 - 1} = \frac{-32}{1} = -32$ teams per round

Between rounds 1 and 2, the number of teams decreases at a rate of 32 teams per round.

2 Calculate the average rate of change between rounds 2 and 3.

Find the rate of change from (2, 32) to (3, 16).

$\frac{16 - 32}{3 - 2} = \frac{-16}{1} = -16$ teams per round

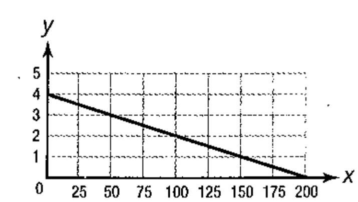
Between rounds 2 and 3, the number of teams decreases at a rate of 16 teams per round.

3 Compare the rates of change.

▶ The rate of change between rounds 1 and 2 is different than it is between rounds 2 and 3. The rate between rounds 2 and 3 is half what it was between rounds 1 and 2.

TRY

Choose a pair of points on the graph below and find the average rate of change between them. Compare your result with those of other students. Did they use the same two points?



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Comparing Average Rates of Change

UNDERSTAND The table below represents the linear function $f(x) = 2x + 1$. Notice that as x-values increase by 1, $f(x)$-values increase by a constant amount, 2. In other words, the function grows by an equal amount, 2, in each unit interval.

+1 +1 +1 +1 +1

x	0	1	2	3	4	5
$f(x)$	1	3	5	7	9	11

+2 +2 +2 +2 +2

A linear function has a constant rate of change. Its average rate of change is the same no matter what interval you are observing. The constant rate of change of a linear function is its **slope**.

An exponential function has a graph that is a curve. An exponential growth function is always increasing, while an exponential decay function is always decreasing. The table below represents an exponential growth function.

+1 +1 +1 +1 +1

x	0	1	2	3	4	5
$f(x)$	1	2	4	8	16	32

×2 ×2 ×2 ×2 ×2

Notice that as each x-value increases by 1, each $f(x)$-value is multiplied by 2. The value of the function, $f(x)$, does not grow by constant amounts over equal intervals, so it does not have a constant rate of change. However, $f(x)$ does grow by the same factor over equal intervals. This function increases by a factor of 2, or doubles, over each unit interval.

The value of an exponential function grows by equal factors over equal intervals. If the factor by which the function changes is greater than 1, then the function represents exponential growth. If the factor is less than 1, then the function represents exponential decay.

The average rates of change for an exponential function grow by the same factor as the values of the function. For the table above, the average rate of change doubled over each unit interval.

+1 +1 +1 +1 +1

x	0	1	2	3	4	5
$f(x)$	1	2	4	8	16	32

+1 +2 +4 +8 +16

×2 ×2 ×2 ×2

Connect

Find and describe the average rate of change for four consecutive pairs of values in the table. What type of function is this?

x	-3	-2	-1	0	1
$f(x)$	64	16	4	1	$\frac{1}{4}$

1

Determine the average rate of change for consecutive pairs of values $(x, f(x))$.

Be sure that the intervals are the same between each pair of points. The difference between each pair of x-values in the table is 1 unit, so the intervals are the same.

between $(-3, 64)$ and $(-2, 16)$:

$$\frac{16 - 64}{-2 - (-3)} = \frac{-48}{1} = -48$$

between $(-2, 16)$ and $(-1, 4)$:

$$\frac{4 - 16}{-1 - (-2)} = \frac{-12}{1} = -12$$

between $(-1, 4)$ and $(0, 1)$:

$$\frac{1 - 4}{0 - (-1)} = \frac{-3}{1} = -3$$

between $(0, 1)$ and $\left(1, \frac{1}{4}\right)$:

$$\frac{\frac{1}{4} - 1}{1 - 0} = \frac{-\frac{3}{4}}{1} = -\frac{3}{4}$$

2

Compare the average rates of change in order to classify the function.

The average rates of change for the first four consecutive pairs of points are: $-48, -12, -3, -\frac{3}{4}$.

These rates are different. The rate of change is not constant, so this is not a linear function.

The value of $f(x)$ decreases by a common factor over each interval.

between $(-3, 64)$ and $(-2, 16)$:

$$\frac{16}{64} = \frac{1}{4}$$

between $(-2, 16)$ and $(-1, 4)$:

$$\frac{4}{16} = \frac{1}{4}$$

between $(-1, 4)$ and $(0, 1)$:

$$\frac{1}{4} = \frac{1}{4}$$

between $(0, 1)$ and $\left(1, \frac{1}{4}\right)$:

$$\frac{\frac{1}{4}}{1} = \frac{1}{4}$$

▶ Since the values of $f(x)$ change by an equal factor over equal intervals and that factor is $\frac{1}{4}$, this is an example of an exponential decay function.

DISCUSS

Why is it important to keep the intervals between each pair of values $(x, f(x))$ the same when comparing average rates of change?

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EXAMPLE A Determine the average rate of change between several consecutive pairs of points for the function $f(x) = -3x + 2$. Describe how the function is changing and classify it.

1 Create a table of ordered pairs for the function.

x	$f(x) = -3x + 2$	$f(x)$
-2	$f(-2) = -3(-2) + 2 = 6 + 2 = 8$	8
-1	$f(-1) = -3(-1) + 2 = 3 + 2 = 5$	5
0	$f(0) = -3(0) + 2 = 0 + 2 = 2$	2
1	$f(1) = -3(1) + 2 = -3 + 2 = -1$	-1
2	$f(2) = -3(2) + 2 = -6 + 2 = -4$	-4

2 Determine the average rate of change for four consecutive pairs of values $(x, f(x))$.

Be sure that the intervals are the same between each pair of points.

between $(-2, 8)$ and $(-1, 5)$:
$\frac{5 - 8}{-1 - (-2)} = \frac{-3}{1} = -3$

between $(-1, 5)$ and $(0, 2)$:
$\frac{2 - 5}{0 - (-1)} = \frac{-3}{1} = -3$

between $(0, 2)$ and $(1, -1)$:
$\frac{-1 - 2}{1 - 0} = \frac{-3}{1} = -3$

between $(1, -1)$ and $(2, -4)$:
$\frac{-4 - (-1)}{2 - 1} = \frac{-3}{1} = -3$

3 Compare the average rates of change.

The average rates of change are all the same, -3.

Since the rate of change is constant, $f(x) = -3x + 2$ must be a linear function.

▶ The rate of change, or slope, is -3 for all pairs of values. The function is linear.

DISCUSS

Does the equation $y = -3x + 5$ provide any clues about what the rate of change for the linear function is? Explain.

EXAMPLE B Compare the rates of change for $f(x) = 10^x$ and function g, which is represented in the table.

x	$g(x)$
−1	$\frac{1}{8}$
0	1
1	8
2	64
3	512

1 Create a table of values for f.

x	$f(x) = 10^x$	$f(x)$
−1	$f(-1) = 10^{-1} = \frac{1}{10}$	$\frac{1}{10}$
0	$f(0) = 10^0 = 1$	1
1	$f(1) = 10^1 = 10$	10
2	$f(2) = 10^2 = 100$	100
3	$f(3) = 10^3 = 1{,}000$	1,000

2 Find the average rate of change for three consecutive intervals for function f.

between (0, 1) and (1, 10): $\frac{10-1}{1-0} = \frac{9}{1} = 9$

between (1, 10) and (2, 100):

$\frac{100-10}{2-1} = \frac{90}{1} = 90$

between (2, 100) and (3, 1,000):

$\frac{1{,}000-100}{3-2} = \frac{900}{1} = 900$

The rate of change of the function f is not constant. Each average rate of change is 10 times the previous rate of change.

3 Find the average rate of change for three consecutive points for function g.

between (0, 1) and (1, 8): $\frac{8-1}{1-0} = \frac{7}{1} = 7$

between (1, 8) and (2, 64):

$\frac{64-8}{2-1} = \frac{56}{1} = 56$

between (2, 64) and (3, 512):

$\frac{512-64}{3-2} = \frac{448}{1} = 448$

The rate of change of the function g is not constant. Each average rate of change is 8 times the previous average rate of change.

▶ The average rates of change for function f are growing more rapidly than the average rates of change for function g.

DISCUSS

By what factor are the values of function $f(x)$ growing? Does the equation $f(x) = 10^x$ help you determine that factor? How could you write an explicit expression for $g(x)$?

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LESSON 8

Writing Linear Equations in Two Variables

UNDERSTAND Sometimes, the relationship between two quantities (such as distance and time) can be modeled by a **linear equation**. In such an equation, each quantity is represented by a different variable. Linear equations in two variables have many of the same characteristics as linear equations with one variable.

Consider the simple situation of filling a pool with water. Suppose that water pours into the pool at a rate of 50 gallons per hour. The equation below describes the relationship between the total number of gallons of water in the pool (y) and the number of hours (x) since filling began.

$y = 50x$

This equation relates two variables, y and x, and it is linear because both variables are to the first power.

UNDERSTAND A linear equation in two variables can be written in **slope-intercept form**, $y = mx + b$, where:

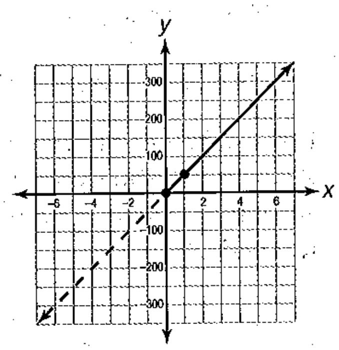
- y and x are variables
- m is the **slope** of the equation's graph, or its rate of change
- b is the ***y*-intercept**, or the y-coordinate where the graph intersects the y-axis

In the situation described above, the equation $y = 50x$ is in slope-intercept form. The graph of the equation will have a slope (m) of 50, the coefficient of x. Rewriting the equation as $y = 50x + 0$ shows that the graph has a y-intercept at (0, 0).

UNDERSTAND A linear equation in two variables can be graphed on a coordinate plane. Each axis of the plane represents one variable. The graph of a linear equation in two variables will be a straight line.

To graph an equation in slope-intercept form, first plot a point at the y-intercept, $(0, b)$. Then use the slope to find a second point. Finish by drawing a line through the two points.

To graph $y = 50x$, first plot a point at (0, 0). Using the slope of 50, or $\frac{50}{1}$, move 50 units up and 1 unit to the right to plot a second point at (1, 50). The line drawn through these points is the graph of the equation.



Connect

A botanist transplanted a plant that was 3 centimeters (cm) tall into an experimental soil. He then took measurements once a week and found an average growth rate of 0.5 cm per week. Write a linear equation in two variables to describe the height of the plant over time. Then graph the equation.

1 Examine the given information.

The two variables are the time since the plant was transplanted and the height of the plant. Time is the independent variable, and the height of the plant is the dependent variable, since the height of the plant depends on how much time has passed.

The plant grows an average of 0.5 cm each week. This is the rate of change, or slope.

At the start, when time equals 0, the plant is 3 cm tall. This is the *y*-intercept.

2 Define variables and write an equation.

Since time is the independent variable, let *x* be the number of weeks since the experiment began.

For the dependent variable, let *y* be the height of the plant in centimeters.

Write a linear equation with a slope of 0.5, or $\frac{1}{2}$, and a *y*-intercept of 3.

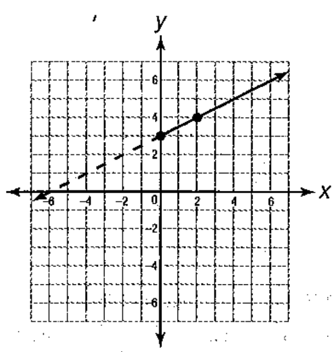
▶ $y = \frac{1}{2}x + 3$

3 Graph the equation.

Plot a point at the *y*-intercept, (0, 3).

Then, using the slope of $\frac{1}{2}$, move two units to the right and one unit up to plot a second point at (2, 4). Draw a line through the points.

▶



DISCUSS

The points (−4, 1) and (1,000, 503) are on the graph of the equation. Are these solutions to the problem?

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EXAMPLE A Tommy has 200 fliers to hand out. He hands out an average of 15 fliers each hour. Write and graph an equation to model the situation.

1 Examine the given information.

The two variables are the time since Tommy started handing out fliers and the number of fliers remaining. Time is the independent variable, and the number of fliers is the dependent variable, since the number of fliers remaining depends on how much time has passed.

Tommy starts with 200 fliers, so this is the *y*-intercept.

He hands out 15 per hour, which is the rate of change. But the number of fliers remaining is decreasing, so the slope should be negative. The slope is −15.

2 Define variables and write an equation.

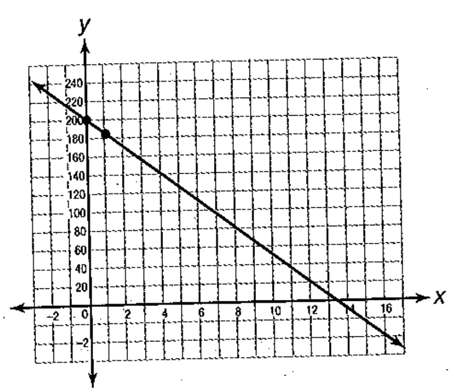
Let *x* be the amount of time that has passed in hours. Let *y* be the number of fliers that Tommy has. Write an equation in slope-intercept form with slope $m = -15$ and *y*-intercept $b = 200$.

▶ $y = -15x + 200$

3 Graph the equation.

Plot a point at the *y*-intercept, (0, 200). Then, use the slope of −15 to plot a second point. From (0, 200), move 1 unit to the right and 15 units down, and plot a point at (1, 185). Draw a line through the two points.

▶



DISCUSS What are the limitations on the variables *x* and *y* in the context of this problem?

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EXAMPLE B A Web site sells MP3 downloads of music albums for $10 and Blu-ray discs of movies for $30. Fiona wants to buy some albums and some movies. She plans to spend a total of $150. Write and graph an equation to represent the situation.

1 Review the given information.

The variables in this situation are the number of albums to be bought and the number of movies to be bought. There is no clear dependent or independent variable, so you can assign the variables either way.

2 Define the variables and write an equation.

Let x be the number of albums to be bought, and let y be the number of movies to be bought.

The amount to be spent on albums is $10x$.
The amount to be spent on movies is $30y$.
The total to be spent must equal 150.

▶ $10x + 30y = 150$

Notice that this equation is not in slope-intercept form.

3 Find two points on the line.

Choose a value to substitute for x in order to find a point on the line. Let's choose 0.

$$10(0) + 30y = 150$$
$$30y = 150$$
$$y = 5$$

The point (0, 5) lies on the line.

You can also choose a number to substitute for y in order to find a point. Again, let's choose 0.

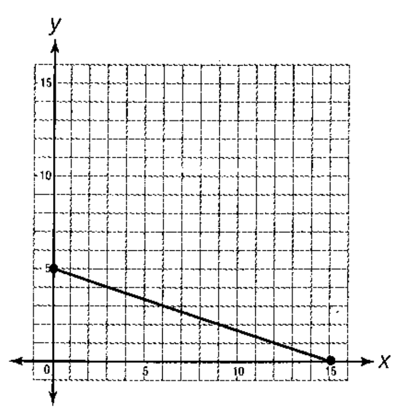
$$10x + 30(0) = 150$$
$$10x = 150$$
$$x = 15$$

The point (15, 0) lies on the line.

4 Graph the line.

Plot the points (0, 5) and (15, 0) and connect them with a line.

▶



DISCUSS

What are the limitations on the variables x and y in the context of this problem?

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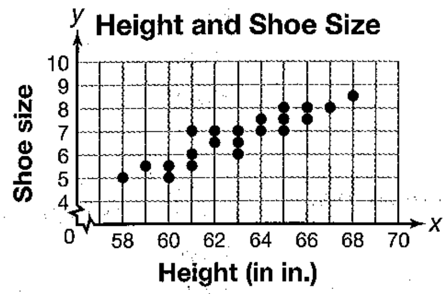
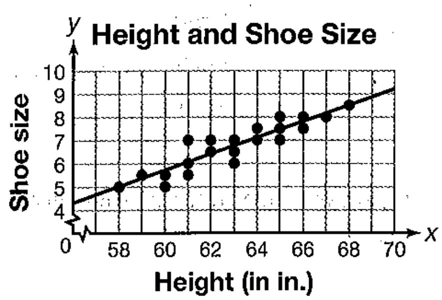
LESSON 29

Constructing and Analyzing Scatter Plots

UNDERSTAND When you study the relationship between two variables—such as the heights and shoe sizes of a group of students—you are working with **bivariate data**. Bivariate data can be written as a set of (x, y) ordered pairs and graphed on a coordinate plane. This kind of graph is called a **scatter plot**. A scatter plot can help you interpret bivariate data. The scatter plot below shows a set of ordered pairs in which the x-values represent heights and the y-values represent shoe sizes.



Look at the shape formed by the plotted points. The shape resembles a straight line. This suggests a linear relationship between the variables. You can draw a line to fit, or model, the data. The line you draw represents a linear function. If the line is a good fit, you can use the graph and the equation of the line to interpret and make predictions about the data.



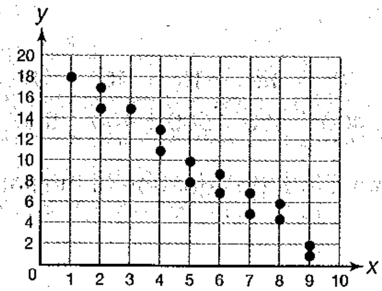
The line appears to be a good fit. The data points slant up from left to right, indicating a positive linear relationship. The line has a positive slope and is close to most data points.

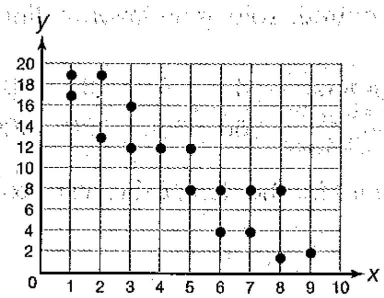
You could also show that the line is a good fit for the data by calculating **residuals**. For each point (x, y) on the scatter plot, there is a corresponding point $(x, \hat{y})$ on the line of fit. A residual is equal to the difference $y - \hat{y}$. Residuals measure the difference of each actual y-value and the expected y-value $(\hat{y})$, which is based on the equation of the line of fit.

Residuals help you determine how accurately the linear function could predict actual points on the scatter plot. That is, if the values of the residuals are relatively small, the linear function is a good fit for the data. So, for any value of x, you could use the equation of the line to make a good prediction about what the value of y would be, and vice versa.

Connect

Draw a line of fit for each of the scatter plots. Determine how well each fits the data.

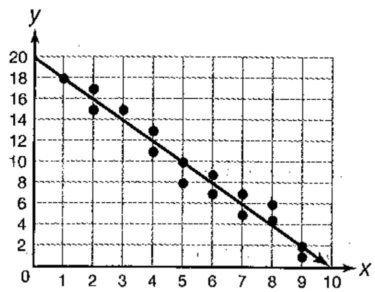


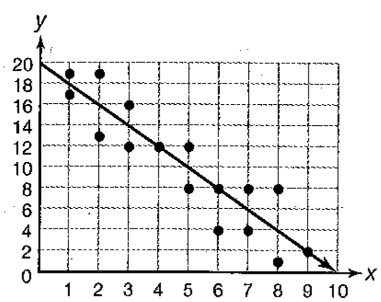


1 Draw a line to model the data for each scatter plot.

For each plot, draw a line that has about as many points above it as below it.

▶





2 Use residuals to determine how well the lines fit the data in the first plot.

Pick several data points, such as (1, 18), (4, 11), (6, 7), and (8, 6). Find the corresponding points, $(x, \hat{y})$, on the line for those x-values: (1, 18), (4, 12), (6, 8), and (8, 4). Calculate the residuals.

(1, 18): $y - \hat{y} = 18 - 18 = 0$

(4, 11): $y - \hat{y} = 11 - 12 = -1$

(6, 7): $y - \hat{y} = 7 - 8 = -1$

(8, 6): $y - \hat{y} = 6 - 4 = 2$

▶ None of the residuals have large values. The line fits the first data set well.

3 Use residuals to determine how well the line fits the data in the second plot.

Pick several data points: (1, 19), (4, 12), (6, 4), and (8, 8). Find the corresponding points on the line: (1, 18), (4, 12), (6, 8), and (8, 4). Calculate the residuals.

(1, 19): $y - \hat{y} = 19 - 18 = 1$

(4, 12): $y - \hat{y} = 12 - 12 = 0$

(6, 4): $y - \hat{y} = 4 - 8 = -4$

(8, 8): $y - \hat{y} = 8 - 4 = 4$

▶ Some of the residuals have large values. The line does not fit the second data set as well.

DISCUSS

Are the lines drawn the only possible lines of fit that could have been drawn for these scatter plots? Why or why not?

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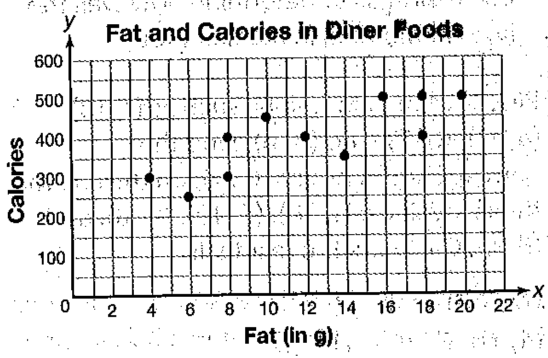
EXAMPLE For a health project, Dylan recorded the number of grams of fat and the number of calories in lunch entrees sold at his favorite diner.

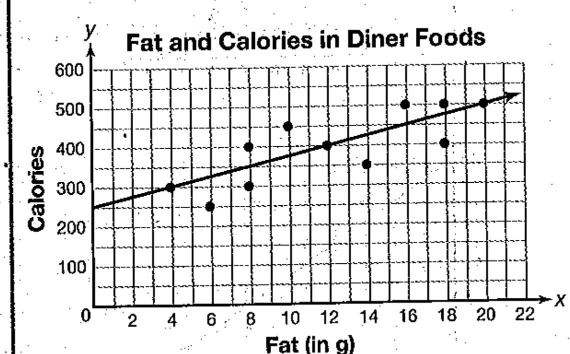
Fat (in grams)	4	6	8	8	10	12	14	16	18	18	20
Calories	300	250	300	400	450	400	350	500	400	500	500

Create a scatter plot for the data. Draw a line to fit the data. Find the equation of the line.

1 Use the ordered pairs of data items to create a scatter plot.



2 Draw a line to fit the data.



3 Write an equation for the line of fit.

The points (4, 300) and (12, 400) are on the line. Use those points to find the slope.

$$m = \frac{400 - 300}{12 - 4} = \frac{100}{8} = 12.5$$

The y-intercept is at (0, 250), so $b = 250$.

▶ The equation of the line is $y = 12.5x + 250$.

DISCUSS

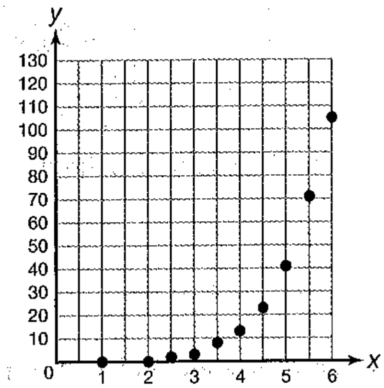
Explain what the slope of the line tells you in this context. Do the data show a positive linear relationship or a negative linear relationship?

LESSON

30 Best Fit and Correlation

UNDERSTAND You can draw a line of fit for a scatter plot by analyzing the data visually. Someone else, however, could look at the same data and draw a slightly different line. To find the line that best fits the data, you need to use a process called regression analysis. Regression analysis helps you find the function that minimizes residuals.

When there seems to be a linear relationship in the data, regression analysis can find the equation of a **line of best fit**. But not all bivariate data show a linear association. In some cases, the relationship between the variables is better modeled by a curve, as in the scatter plot shown. For data that do not have a linear association, you will need to find a **curve of best fit**. To find the equation of either a line of best fit or a curve of best fit, you can use a graphing calculator to perform a regression analysis.



UNDERSTAND Once you have determined the line of best fit for bivariate data, you can use the **correlation coefficient**, r, to describe the strength and direction of the relationship between the two variables.

These statements will help you interpret a correlation coefficient.

- The value of r is always in the range $-1 \leq r \leq 1$.
- If r is close to 1, the data show a strong positive correlation
- If r is close to -1, the data show a strong negative correlation.
- If $r = 0$, the data do not show a linear correlation.

When bivariate data have a strong correlation, the predictions you make by using the line of best fit are likely to be very accurate. When there is a weak correlation, these predictions will tend to be less accurate. A positive correlation means that as one variable increases, the other variable tends to increase also. A negative correlation means that as one variable increases, the other tends to decrease.

The correlation coefficient, r, is calculated using a rather complex formula involving residuals. Fortunately, you can use a calculator to do that work for you!

Keep in mind that there is a crucial difference between correlation and causation. A strong correlation does not tell you that x is the cause of y. For example, buying lemonade and going to the beach might be strongly correlated, but one does not cause the other.

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Connect

On a graphing calculator, create a scatter plot and draw a curve of best fit for the data in the table below.

Time, x (in minutes)	0	2	4	6	8	10
Number of Bacteria, y	5	11	25	57	130	290

1 Enter the data into your calculator.

Press **STAT**. Then select **1: Edit**.

Use L1 for time and L2 for number of bacteria. Enter each ordered pair in the table, as shown below.

L1	L2	L3 2
0	5	
2	11	
4	25	
6	57	
8	130	
10	290	
------	------	

L2(1)=5

2 Perform an exponential regression.

Press **STAT**.

Move the cursor to the **CALC** menu. Then select **0:ExpReg**. Press **ENTER**.

ExpReg
y=a*b^x
a=4.934294253
b=1.503276798
r^2=.999954037
r=.9999770182

Rounding a and b, the equation for the curve of best fit is $y = 4.93(1.5)^x$.

3 Create a scatter plot.

To show scatter plots on your calculator, turn the STAT PLOT on. Press **2nd** **Y=**.

Select **1: Plot 1**. Press **ENTER**.

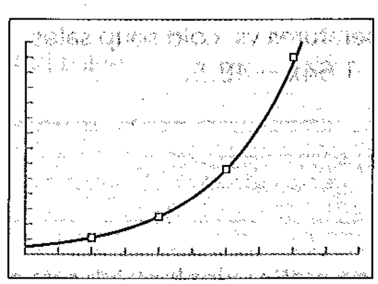
If **Off** is highlighted, move the cursor to highlight **On**. Press **ENTER**.

Then, press **ZOOM**. Choose **9: ZoomStat**.

4 Graph the curve of best fit.

Press **Y=** and for Y_1 enter 4.93(1.5)^X.

Then press **GRAPH**.



CHECK

On your graphing calculator, press **2nd** **GRAPH**. This brings up a table for $y = 4.93(1.5)^x$. Compare your calculator table to the table above. Does it show that the curve is a good fit for the data? Is the fit perfect?

EXAMPLE The table on the right shows the daily high temperatures on six days and the number of cups of hot soup and of cold soup that were sold at a soup stand on each day.

Find a line of best fit to model the relationship between high temperatures and hot soup sales. Find another line of best fit to model the relationship between high temperatures and cold soup sales. Compare the lines.

Daily High Temperature	Cups of Hot Soup Sold	Cups of Cold Soup Sold
30	85	1
40	70	16
50	45	35
60	22	52
70	12	60
80	2	86

1 Enter the data in your calculator.

Press STAT. Then select **1: Edit**. Enter daily high temperatures in L1, hot soup sales in L2, and cold soup sales in L3.

2 Perform a linear regression for high temperatures (L1) and hot soup sales (L2).

Press STAT. Move the cursor to the **CALC** menu. Select **4:LinReg(ax+b)**. Press ENTER.

```
LinReg
 y=ax+b
 a=-1.748571429
 b=135.5047619
 r²=.972959429
 r=-.9863870908
```

▶ The equation modeling high temperatures vs. hot soup sales is $y = -1.75x + 135.5$.

3 Perform a linear regression for high temperatures (L1) and cold soup sales (L3).

Press STAT. Move the cursor to the **CALC** menu. Then select **4:LinReg(ax+b)**. Press ENTER. Now press 2nd STAT. Select **1:L1**. Press , . Then press 2nd STAT again. Select **3:L3**. Press ENTER.

```
LinReg
 y=ax+b
 a=1.64
 b=-48.53333333
 r²=.9877168439
 r=.9938394457
```

▶ The equation modeling high temperatures vs. cold soup sales is $y = 1.64x - 48.5$.

4 Compare the lines.

The line for hot soup sales shows a negative correlation. The line for cold soup sales shows a positive correlation.

The correlation coefficients are close to −1 and 1, respectively. The lines are good fits for the data.

MODEL

Create a scatter plot for both sets of data. Then graph the lines of best fit. Compare how well the lines fit the data.

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